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FUNDAMENTAL SOLUTION OF THE INTERNAL-WAVE EQUATION FOR A MEDIUM WITH A DISCONTINUOUS BRUNT–VÄISÄLÄ FREQUENCY†

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A two-layer stratified medium with constant, but different, values of the Brunt–Väisälä (BV) frequencies in the layers is considered. An integral representation of the fundamental solution of the internal-wave equation is constructed in the Boussinesq approximation. The wave patterns in the upper and lower layers are investigated under the assumption that the source is located in the lower layer. It is shown that the fundamental solution in the lower layer is expressed in terms of the same standard functions as in the case of a single layer. The situation is more complicated in the upper layer and the investigation is based on a study of the branching points of a certain multivalued function. For long times, approximate solutions are obtained by the stationary-phase method. The limiting case when the BV frequencies in the layers are only slightly different is investigated. In another limiting case, a surface exists (a circular cone or a cylinder, depending on the ratio of the BV frequencies in the layers) close to which the formulae obtained using the stationary-phase method are inapplicable. The special asymptotic form of the fundamental solution on this surface is calculated. © 1997 Elsevier Science Ltd. All rights reserved.

1. INTEGRAL REPRESENTATION OF THE FUNDAMENTAL SOLUTION

We consider the function $G(x, y, z, z_0, t)$ which is a generalized solution of the equation

$$\frac{\partial^2}{\partial t^2}(\Delta G)+N^2(z)\Delta_2 G=-\delta(x)\delta(y)\delta(z+z_0)\delta(t)$$
$$N(z)=N_1\theta(z)+N_2\theta(-z),\quad \Delta_2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2},\quad \Delta=\Delta_2+\frac{\partial^2}{\partial z^2} \tag{1.1}$$

where δ is the delta-function and $\theta(z)$ is the Heaviside function, which is equal to zero when $z \leqslant 0$ and equal to unity when $z > 0$. We know [1] that G is the solution of the homogeneous equation (1.1) with initial conditions

$$G|_{t=0}=0,\quad \left.\frac{\partial G}{\partial t}\right|_{t=0}=\frac{1}{4\pi\sqrt{r^2+(z+z_0)^2}},\quad r^2=x^2+y^2$$

It is assumed that $z_0 > 0$, $N_1 \neq N_2$. A general solution of Eq. (1.1) will be sought which is bounded at infinity and is a continuously differentiable function of (x, y, z) in the whole of space with a deleted point $(0, 0, -z_0)$. The small vertical displacements of the fluid particles, which are induced by the sources and dipoles which have begun to function at the initial instant of time, are expressed in terms of derivatives of the function G, while the solution of the general Cauchy problem for the internal wave equation is expressed in terms of a convolution with the fundamental solution.

The fundamental solution has been investigated in detail for N = const. An extensive bibliography can be found, for example, in [2]. New approximate representations for the fundamental solution were investigated in [3]. An investigation of internal waves for the case of a discontinuous BV frequency is of considerable interest for the physics of the atmosphere and ocean. Free oscillations in such a medium have been studied, but forced oscillations have not been investigated to any great extent because of the mathematical difficulties involved, since fairly detailed estimates of multivalued analytical functions are required in order to study the formal integral representation of the fundamental solution.

We will use the well-known integral representations for δ-functions [1]

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$$\delta(x)\delta(y)=\frac{1}{2\pi}\int_0^{+\infty} uJ_0(ru)du,\quad \delta(t)=\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}\exp(pt)dt,\quad \beta>0 \tag{1.2}$$

and will seek the fundamental solution of Eq. (1.1) in the form

$$G=\frac{1}{(2\pi)^2}\int_0^{+\infty}\int_{\beta-i\infty}^{\beta+i\infty} uJ_0(ru)G^*(u,p,z)\exp(pt)dudp \tag{1.3}$$

Substituting (1.3) into (1.2) and using (1.1), we obtain the following equation for determining G^*

$$p^2\left(\frac{d^2G^*}{dz^2}-u^2G^*\right)-N^2(z)u^2G^*=-\delta(z+z_0) \tag{1.4}$$

Green's function for the ordinary differential equation (1.4) with constant coefficients can be found in an elementary way as:
for $z>0$

$$G^*=-\frac{4\pi}{up(K_1+K_2)}\exp\left\{-\frac{u}{p}(zK_1+z_0K_2)\right\},\quad K_i^2=p^2+N_i^2 \tag{1.5}$$

for $z<0$

$$G^*=-\frac{2\pi(K_2-K_1)}{2upK_2(K_2+K_1)}\exp\left(-\frac{u}{p}K_2(z-z_0)\right)-\frac{2\pi}{upK_2}\exp\left(-\frac{u}{p}K_2(z+z_0)\right) \tag{1.6}$$

Substituting (1.5) and (1.6) into (1.3) and making use of the fact that

$$\int_0^{+\infty} J_0(ru)\exp(-\alpha u)du=\frac{1}{\sqrt{\alpha^2+r^2}},\quad \alpha>0$$

we obtain, for $z>0$

$$-4\pi^2 iG=\frac{1}{(N_2^2-N_1^2)}\int_{\beta-i\infty}^{\beta+i\infty}\frac{(K_2-K_1)\exp(pt)dp}{\sqrt{p^2r^2+(zK_1+z_0K_2)^2}} \tag{1.7}$$

Similarly, substituting (1.6) into (1.3), we obtain, for $z<0$

$$-8\pi^2 iG=\int_{\beta-i\infty}^{\beta+i\infty}\frac{\exp(pt)dp}{K_2\sqrt{p^2r^2+K_2^2(z+z_0)^2}}-\int_{\beta-i\infty}^{\beta+i\infty}\frac{(K_2-K_1)\exp(pt)dp}{K_2(K_2+K_1)\sqrt{p^2r^2+K_2^2(z-z_0)^2}} \tag{1.8}$$

2. WAVE PATTERN IN THE LOWER HALF-SPACE

We know that the fundamental internal-wave equation for a constant BV frequency has the form

$$G=\frac{1}{4\pi N\sqrt{r^2+z^2}}\Phi\left(Nt,\frac{z}{\sqrt{r^2+z^2}}\right)$$

and different integral representations and asymptotic formulae (see [2, 3], for example) have been obtained for the function $\Phi(\tau,\lambda)$. In particular

$$\Phi(\tau,\lambda)=\frac{1}{2\pi}\int_{-\infty-i0}^{+\infty-i0}\frac{\exp(i\tau\nu)d\nu}{\sqrt{(\nu^2-1)(\nu^2-\lambda^2)}}=\frac{2}{\pi}\operatorname{sign}(1-\lambda)\int_\lambda^1\frac{\sin(\tau\nu)d\nu}{\sqrt{|(1-\nu^2)(\nu^2-\lambda^2)|}}$$

If we put

$$R_{\pm}^2 = r^2 + (z \pm z_0)^2, \quad \lambda_{\pm} = \frac{|z \pm z_0|}{R_{\pm}} \tag{2.1}$$

in (1.8) and make a change of variables by putting

$$p = iN_2\nu, \quad \tau = N_2 t, \quad \mu^2 = N_1^2 / N_2^2 = 1 + \delta \tag{2.2}$$

then, when $z < 0$, this formula becomes

$$4\pi N_2 G = \frac{\Phi(\tau,\lambda_-)}{R_-} - \frac{\Phi(\tau,\lambda_+)}{R_+} +$$
$$+ \frac{1}{(1-\delta^2)R_-}\left\{\left(\frac{d^2}{d\tau^2}+1\right)\Phi(\tau,\lambda_-) - \frac{1}{\mu}\left(\frac{d^2}{d\tau^2}+\mu^2\right)\Phi\left(\mu\tau,\frac{\lambda_-}{\mu}\right)\right\} \tag{2.3}$$

Note that the first two terms describe perturbations from two sources, which are symmetrical about the $z = 0$ plane in a medium with a constant BV frequency N_2, while the third term describes the effect from the presence of a discontinuity in the BV frequency and the result of interaction of the sources.

Since different representations in the form of integrals and series and different asymptotic expressions are known for the function, formula (2.3) is a source for obtaining approximations of the fundamental solution in the lower half-space. In the upper half-space the fundamental solution is no longer expressed in terms of the standard function $\Phi(\tau, \lambda)$. This complicates the investigation, which we shall carry out in the next section.

We will now find the simplest asymptotic formula for the fundamental solution in the lower half space. As we know, the asymptotic formula

$$\Phi(\tau,\lambda) = \Phi_1\left(\tau - \frac{\pi}{4},\lambda\right) + \Phi_1\left(\lambda\tau + \frac{\pi}{4},\lambda\right)$$
$$\Phi_1(\tau,\lambda) = \sqrt{\frac{2}{\pi}}\frac{\sin\tau}{\sqrt{\tau|1-\lambda^2|}}$$

holds as $\tau \to \infty$ and for non-negative values of the parameter λ and for values of this parameter which are not close to zero or to unity.

On further introducing the function

$$\Phi_2(\tau,\lambda) = \sqrt{\frac{2}{\pi}}\frac{\cos\tau}{\sqrt{\tau^3|1-\lambda^2|}}$$

substituting the expressions for the function $\Phi(\tau, \lambda)$ in terms of the functions $\Phi_1(\tau, \lambda)$, $\Phi_2(\tau, \lambda)$ into formula (2.3) and taking account of (2.2), we obtain the asymptotic formula when $z < 0$, $\tau \to \infty$

$$4\pi N_2 G = \frac{1}{R_-}\left(\Phi_1\left(N_2 t - \frac{\pi}{4},\lambda_-\right) + \Phi_1\left(N_2\lambda_- t + \frac{\pi}{4},\lambda_-\right)\right) -$$
$$-\frac{1}{R_+}\left(\Phi_1\left(N_2 t - \frac{\pi}{4},\lambda_+\right) + \Phi_1\left(N_2\lambda t_+ + \frac{\pi}{4},\lambda_+\right)\right) +$$
$$+\frac{1}{(1-\delta^2)R_-}\left(\Phi_2\left(N_1 t - \frac{\pi}{4},\lambda_-\frac{N_2}{N_1}\right) - \Phi_2\left(N_2 t - \frac{\pi}{4},\lambda_-\right)\right) +$$
$$+\frac{1-\lambda_-^2}{(1-\delta^2)R_-}\left(\Phi_1\left(N_2\lambda_- t + \frac{\pi}{4},\lambda_-\right) - \Phi_1\left(N_1\lambda_- t + \frac{\pi}{4},\lambda_-\frac{N_2}{N_1}\right)\right) \tag{2.4}$$

The quantities $R_{\pm}$, $\lambda_{\pm}$ are defined by formulae (2.1).

Note that terms of the order of $\tau^{-1/2}$ are retained in the first two terms for fixed values of $\lambda_{\pm}$ which differ from zero and unity, while terms of the order of $\tau^{-3/2}$ are retained in the third term in formulae

(2.4). This is associated with the non-uniformity of the asymptotic representation with respect to the parameters N_1 and N_2. If, for a fixed τ, one takes the limit in formula (2.4) when $N_1 \to N_2$, the third and fourth terms become of the order of $1/\sqrt{\tau}$ and the correct asymptotic formula when $\tau \to \infty$ for the case of a constant BV frequency is obtained.

3. THE WAVE PATTERN IN THE UPPER HALF-SPACE

On making the change of variables (2.2) in Eq. (1.7) and using the notation of (2.1), we obtain

$$G = \frac{1}{4\pi^2 N_2 \delta i} \int_{-\infty - i0}^{+\infty - i0} F(\nu)\exp(i\tau\nu)d\nu$$
$$F(\nu) = \frac{\sqrt{\mu^2 - \nu^2} - \sqrt{1-\nu^2}}{\sqrt{B(\nu)}}, \quad B(\nu) = \nu^2 r^2 - (z\sqrt{\mu^2 - \nu^2} + z_0\sqrt{1-\nu^2}) \tag{3.1}$$

It follows from (2.2) that, when $N_2 < N_1$, the parameter $\delta > 1$. The treatment can henceforth be confined solely to this case since, when $N_2 > N_1$, instead of the change (2.2), we can make the change

$$p = iN_1 t, \ \ \tau = N_1 t, \ \ \mu^2 = N_2^2 / N_1^2 = 1 + \delta$$

which leads to the formula which is obtained from (3.1) by permuting N_2 and N_1 and z and z_0.

The investigation was based on a study of the branching points of the integrand. The branch of the function $\sqrt{(1-\nu^2)}$ is regular in the plane with cuts along the real axis joining the points $+1$ and -1 with infinity and takes a value of 1 in the case of a zero value of the argument. The branch of the function $\sqrt{(\mu^2 - \nu^2)}$ is also selected in a similar way. There are further branching points in the case of the integrand which are zeroes of the function $B(\nu)$. If we put

$$\sqrt{\frac{\mu^2 - \nu^2}{1-\nu^2}} = w + \mu \tag{3.2}$$

then

$$\nu^2 = \frac{w^2 + 2\mu w}{\delta + w^2 + 2\mu w}, \quad B(\nu) = \frac{1}{\delta}(1-\nu^2)T(w)$$
$$T(w) = aw^2 + 2bw - \delta c, \quad a = r^2 - \delta z^2, \quad b = \mu a - \delta z z_0, \quad c = (\mu z + z_0)^2 \tag{3.3}$$

$$D = b^2 + \delta ac = (\mu^2 a + \delta z_0^2)(a + \delta z^2) = r^2(\mu^2 r^2 + \delta(z_0^2 - \mu z^2))$$

When $r^2 > \delta\mu^{-2}(\mu^2 z^2 - z_0^2)$, the quadratic trinomial $T(w)$ has two real roots

$$w_1 = \frac{-b + \sqrt{D}}{a}, \quad w_2 = \frac{-b - \sqrt{D}}{a} = \frac{c\delta}{\sqrt{D} - b} \tag{3.4}$$

Using formula (3.3), we obtain

$$\nu_i^2 = 1 - \frac{\delta}{\delta + w_i^2 + 2\mu w_i} = 1 - \frac{a^2}{a^2 + A \pm 2zz_0\sqrt{D}}$$
$$A = ac - 2zz_0 b = a(\mu^2 z^2 + z_0^2) + 2\delta z^2 z_0^2; \quad i = 1,2 \tag{3.5}$$

By using (3.3)–(3.5) and the identity $A^2 - 4z^2 z_0^2 D = a^2(z_0^2 - \mu^2 z^2)^2$, it can be shown that, when

$D > 0$, the sign of the expression $A \pm 2zz_0\sqrt{D}$ is identical with the sign of A and $0 < \nu_2^2 < \nu_1^2 < 1$.

The function $B(\nu)$ has two real zeros $\pm\nu_1$ when $a > 0$, $\delta > 0$ but does not have zeros when $a < 0$, $\delta > 0, D > 0$. When $D > 0$, the zeros of the function $B(\nu)$ are complex conjugates and have a non-zero imaginary part.

Actually, when $a > 0$, $\delta > 0$, we have $D > 0$ and $0 < \nu_2 < \nu_1 < 1$. On account of the choice of the branches in formula (3.2), it is necessary and sufficient that the inequality $w_i + \mu > 0$ be satisfied, in order that the numbers $\pm\nu_i$ should be zeros of the function $B(\nu)$. If $a > 0$, then $w_1 + \mu > 0$, $w_2 + \mu < 0$ and the points $\pm\nu_1$ are therefore zeros of the function $B(\nu)$ while the points $\pm\nu_2$ are not. If, however, $a < 0$, then $w_1 + \mu < 0$, $w_2 + \mu < 0$, so that there are no zeros in this case.

Remark. The inequality $D < 0$ when $\delta > 0$ is satisfied inside the hyperboloid of rotation

$$z^2 - r^2/\delta = z_0^2\mu^{-2} \tag{3.6}$$

It will next be shown that terms, which exponentially tend to zero when the time tends to infinity, correspond to complex zeros.

We now transform formula (3.1). Suppose that $r > z\sqrt{(\delta)}$, $\delta > 0$. In this case, $D > 0$ and the function $F(\nu)$ has, in addition to the branching points ± 1, $\pm\mu$, a further two branching points $\pm\nu_1$, $0 < \nu_1 < 1$, which are zeros of the function $B(\nu)$. We now cut the complex plane along the segments $[\nu_1, \mu]$, $[-\mu, -\nu_1]$. The function $F(\nu)$ is regular in a plane with such cuts. On passing around the cut along a contour which encompasses it the numerator and the denominator of the fraction $F(\nu)$ change sign, and this function is positive on the lower edge of the right-hand cut. The equalities

$$F(-\nu \pm i0) = -\overline{F(\nu \pm i0)}, \quad F(\nu + i0) = \overline{F(\nu - i0)}, \quad \nu \in [\nu_1, \mu] \tag{3.7}$$

hold and, by using these and applying Cauchy's theorem, we transform integral (3.1) to an integral along the segment $[\nu_1, \mu]$. On subdividing this segment into two $[\nu_1, 1]$, $[1, \mu]$ and using the equality (3.7), we obtain

$$G = -\frac{1}{\pi^2 N_2 \delta}(G_1 + G_2), \quad \delta > 0, \quad r > z\sqrt{\delta} \tag{3.8}$$

$$G_1 = \int_{\nu_1}^{1} F(\nu)\sin(\tau\nu)d\nu, \quad G_2 = \int_{1}^{\mu} F(\nu)\sin(\tau\nu)d\nu$$

If $r < z\sqrt{(\delta)}$, the function $B(\nu)$ does not vanish and it is necessary to put $G_1 = 0$ in formula (3.12).

If a point (x, y, z) lies in those domains where $D < 0$, the function $B(\nu)$ does not have more than two pairs of complex conjugate roots and integrals along the vertical cuts in the upper half-plane joining a branching point to infinity, which decrease exponentially as $t \to \infty$, appear in formula (3.8). Consequently, it is not necessary to take account of these terms when deriving approximate formulae using the stationary-phase method.

We will now find the asymptotic form of the fundamental solution in the upper half-plane. We use formulae (3.7)–(3.8) when $\delta > 0$, $r > z\sqrt{(\delta)}$. The function $F(\nu)$, which is defined by formula (3.1), has three branching points in the segment $[\nu_1, \mu]$: $\nu_1 < 1$, 1, $\mu > 1$. We put

$$-B(\nu) = (\nu^2 - \nu_1^2)\chi(\nu), \quad \chi(\nu) > 0, \quad \nu \in [\nu_1, 1] \tag{3.9}$$

where

$$\begin{gathered}\chi(\nu_1) = r^2 + z^2 + z_0^2 + zz_0\left(w_1 + \mu + \frac{1}{w_1 + \mu}\right)\\ -B(1) = r^2 - z^2\delta = a, \quad -B(\mu) = \mu^2 r^2 + z_0^2\delta > 0\end{gathered} \tag{3.10}$$

We now give an estimate of the function defined by formula (3.8) using standard estimates of the stationary-phase method. When $\delta > 0$, we have

$$G = \frac{\xi}{\pi N_2 \delta} \frac{\sqrt{\mu^2 - \nu_1^2} - \sqrt{1 - \nu_1^2}}{\sqrt{2\pi\tau\nu_1\chi(\nu_1)}} \sin\left(\tau\nu_1 + \frac{\pi}{4}\right) -$$

$$-\frac{1}{\pi N_2 \delta} \frac{1}{\sqrt{2\pi\tau^3}} \left(\frac{\sqrt{\mu}\cos(\mu\tau - \pi/4)}{\sqrt{\mu^2 r^2 + z_0^2 \delta}} - \frac{\cos(\tau + (1-2\xi)\pi/4)}{\sqrt{r^2 - z^2\delta}} \right) \tag{3.11}$$

$$\xi = \theta(r^2 - z^2\delta)$$

The term containing $\tau^{-3/2}$ is retained with the aim that formula (3.11) should give a correct approximation in the case of small values of δ.

Note that formula (3.11) is inapplicable for small values of the parameter $a = r^2 - z^2\delta$. More precise estimates (see [4]) are required in this case. The asymptotic form when $a = 0$ is calculated in Section 5.

4. APPROXIMATION FOR SMALL VALUES OF δ

If the BV frequencies in the upper and lower layers only differ by a small amount, this parameter will be small. We will express all the remaining parameters in terms of the small parameter, retaining terms of the zeroth and first orders. When $r > z\sqrt{(\delta)}$, $\delta > 0$, from formulae (3.2), (3.3) and (3.5) we obtain the following expression for the first term in formula (3.11)

$$\frac{1}{2\pi N_2 k R \sqrt{2\pi\tau\lambda}} \left(1 - \frac{\delta}{4\mu R_\mu}\lambda\right) \sin\left(\tau\left(\lambda + \frac{\delta}{2\mu}\frac{z}{R_\mu}\lambda\right) + \frac{\pi}{4}\right)$$

$$R_\mu^2 = r^2 + (\mu z + z_0)^2, \quad R^2 = r^2 + (z + z_0)^2, \quad \lambda = \frac{|\mu z + z_0|}{R_\mu}, \quad k = \frac{r}{R_\mu}$$

5. SMALL VALUES OF THE PARAMETER a

We now treat formulae (3.2), (3.3) and (3.8)–(3.10) when $\delta > 0$, retaining only terms of the zeroth and first order. It can be shown that G_1 is uniformly of the order of a^2 with respect to τ. Using the last equality of (3.8), we write the function G_2 in the form

$$G_2 = \mathrm{Re}\left\{ \int_1^\mu \frac{\sqrt{\mu^2 - \nu^2} - i\sqrt{\nu^2 - 1}}{\sqrt{-B(\nu)}} \sin(\tau\nu) d\nu \right\}$$

$$-B(\nu) = a(w - w_1)(w - w_2)(1 - \nu^2) = (a\sqrt{\mu^2 - \nu^2} - ib\sqrt{\nu^2 - 1})(\sqrt{\mu^2 - \nu^2} - i(w_2 + \mu)\sqrt{\nu^2 - 1}) \tag{5.1}$$

The difficulty in investigating integral (5.1) for small values of a lies in the fact that the branching point ν_1 becomes close to the end point $\nu = 1$. The situation when the stationary point is close to the end point and the asymptotic form is expressed in terms of Fresnel functions has been considered earlier in [4]. In the more complex case being considered, the asymptotic form is no longer expressed in terms of known special functions.

We confine ourselves to the case when $a = 0$. The point $\nu = 1$ makes the greatest contribution to the asymptotic form. According to the general technique for obtaining asymptotic estimates of integrals of this type [4], using formula (3.8) we conclude that the asymptotic formula

$$G = -\frac{1}{2\pi N_1}\left(\frac{2}{\delta\tau}\right)^{3/4} \frac{\Gamma(3/4)}{\sqrt{z z_0}} \sin\left(\tau + \frac{3\pi}{8}\right), \quad a = 0, \quad \tau \to \infty$$

holds.

A study of the fundamental solution for small, but non-zero, values of the parameter a requires further investigation.

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